# Module 3

## Section 1: Loan Repayment – Amortization Method

### Terminology and Notation:

$L$ – Loan Amount
$i$ – Periodic Effective Loan Interest Rate
$n$ – Number of Periodic Payments
$R_k$ – Amount of the $k^{\text{th}}$ Payment

> $I_k$ – Amount of the $k^{\text{th}}$ payment that pays interest on the loan
> $P_k$ – Amount of the $k^{\text{th}}$ payment that repays principal
>
> NOTE: $R_k = P_k + I_k$ $\qquad L = \sum_{k=1}^{n} P_k$

$B_k$ – Balance Immediately After the $k^{\text{th}}$ Payment

### Determining Loan Balances

We can determine loan balances in two ways:

$B_k \overset{Ret}{=} AV(L) - AV(\text{Past Payments})$ $\qquad$ *Ret* – Retrospective

$B_k \overset{Pro}{=} PV(\text{Remaining Payments})$ $\qquad$ *Pro* – Prospective

### Determining the amount of interest and/or the amount of principal in each payment:

Use the following:

$$I_k = i \cdot B_{k-1} \qquad P_k = R_k - I_k$$

### Important Remarks:

1. Amount of principal repaid during a period, say from time $k$ to time $m$ ($k < m$)
   = balance at end of period – balance at beginning of period
   $= \sum_{i=k+1}^{m} P_i = B_k - B_m$

2. Amount of interest paid during a period, say from time $k$ to time $m$ ($k < m$)
   = total amount paid during period – amount of principal repaid during period
   $= \sum_{i=k+1}^{m} R_i - (B_k - B_m)$

**Basic Relationships for Level Payment** ($R_k = R$ for all $k$) **Amortizations**:

$L = B_0 = Ra_{\overline{n}|i}$

$B_k \overset{Pro}{=} Ra_{\overline{n-k}|i} \overset{Ret}{=} L(1+i)^k - Rs_{\overline{k}|i}$

$I_k = i \cdot B_{k-1} = i \cdot Ra_{\overline{n-(k-1)}|i} = R(1 - v^{n-k+1})$

$P_k = R - I_k = Rv^{n-k+1}$

These relationships are captured in a **Level Payment Loan Amortization Table**:

| Time | Payment | Interest Paid | Principal Repaid | Balance (Outstanding Principal) |
|---|---|---|---|---|
| 0 | | | | $L = B_0 = Ra_{\overline{n}|}$ |
| 1 | R | $I_1 = R(1 - v^n)$ | $P_1 = Rv^n$ | $B_1 = Ra_{\overline{n-1}|}$ |
| 2 | R | $I_2 = R(1 - v^{n-1})$ | $P_2 = Rv^{n-1}$ | $B_2 = Ra_{\overline{n-2}|}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $k$ | R | $I_k = R(1 - v^{n+1-k})$ | $P_k = Rv^{n+1-k}$ | $B_k = Ra_{\overline{n-k}|}$ |

**Remarks about this table**:

1. $\{P_1, P_2, \cdots, P_n\}$ is a geometric sequence with common ratio $r = 1 + i$.

2. $L = \sum_{k=1}^{n} P_k = P_1 + P_2 + \cdots + P_n = P_1[1 + (1+i) + \cdots + P_1(1+i)^n] = P_1 s_{\overline{n}|}$

3. We can relate the balance at time $k$ to the balance at time $m$ ($k < m$) as follows:

   $B_k = Ra_{\overline{m-k}|} + B_m v^{m-k}$ (Note that this is a one-step TVM calculation.)

   As written, this equation has a valuation date at time $k$. Multiplying both sides by $(1+i)^{m-k}$ and rearranging terms gives the time $m$ equation $B_m = B_k(1+i)^{m-k} - Rs_{\overline{m-k}|}$. With $k = 0$, this is the prospective method of determining the balance.

4. As a special case of the previous remark, we can calculate balances at neighboring times in two ways:

$$B_{k+1} = B_k(1+i) - R$$

   or

$$B_{k+1} = B_k - P_{k+1}$$

**Module 3 Section 1 Problems:**

1. A 10-year loan of 5000 at an annual effective interest rate of 6% is amortized with monthly payments. Determine the amount of the monthly payments.

2. A 20-year loan of 10000 is repaid with quarterly payments of 334.47. Determine the nominal interest rate compounded quarterly charged by the lender.

3. A 30-year mortgage of 200,000 is amortized with monthly payments using a nominal interest rate of 6% compounded monthly.

   (a) Determine the total amount of interest paid on the loan.
   (b) Determine amount of principal repaid during the third 3-year period.
   (c) Determine the amount of interest paid during the 1st year.
   (d) Determine the amount of interest paid during the 29th year.

4. An *n*-year loan of 30000 at 9% interest compounded quarterly is repaid with quarterly payments of 1000 plus an additional final payment.

   (a) Determine the amount of the final payment if it is larger than 1000.
   (b) Determine the amount of the final payment if it is smaller than 1000.

5. A lender charges a nominal interest rate of 3% compounded monthly on a 10-year loan. The amount of principal repaid in the 12th payment is 334.05. Determine the amount borrowed.

6. The lender of a 50,000 loan charges a periodic effective interest rate of 5.06%. The periodic payments are non-level and continue as long as necessary in order to pay off the loan, with the first payment equal to 2000 and subsequent payments increasing by 2% over their previous payments.

   (a) Determine the outstanding principal immediately after the 5th payment.
   (b) Determine the loan balance immediately before the 6th payment.
   (c) Determine the amount of interest paid in the 10th installment.
   (d) Determine the amount of principal repaid in the 10th payment.

7. A 20-year loan at an annual effective interest rate of 4% is repaid with increasing annual payments. The first payment is equal to 1000 and subsequent payments increase by 200 over their previous payments.

   (a) Determine the outstanding principal immediately after the 4th payment.
   (b) Determine the amount of interest paid in the 9th payment.
   (c) Determine the amount of interest paid during the second 4-year period.

**Solutions to Module 3 Section 1 Problems:**

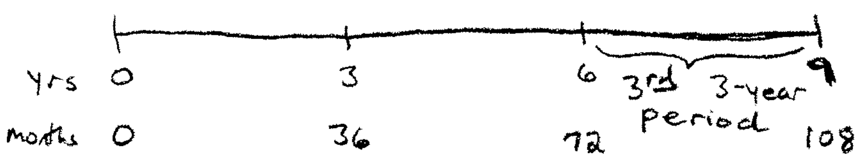
1. Since the payments are monthly, convert the 6% aeir to its equivalent meir, $i$. We get $i = 1.06^{1/12} - 1$. There are 120 monthly payments, and so our basic equation is $5000 = Ra_{\overline{120}|i}$ which by using TVM gives $R = 55.11$.

2. There are 80 quarterly payments of 334.47 that have a present value of 10000 when using a qeir of $i$. We have $10000 = 334.47a_{\overline{80}|i}$ which by using TVM gives $i = .0304$. Then the nominal interest rate compounded quarterly is $i^{(4)} = 4i = 12.16\%$.

3. First determine the amount of each monthly payment: $200000 = Ra_{\overline{360}|0.005}$ and so to the nearest penny, $R = 1199.10$.

   (a) The total amount of interest paid on the loan is $360R - 200000 = 231{,}676$. Note that this is just a dollar amount and does not reflect the time value of money. The problem does not ask for the *present value* or *accumulated value* of the interest payments, and so there is no valuation date. The problem only asks for the total amount of interest paid, and so we just add up the dollar amounts.

   (b) The third 3-year period extends from time 72 to time 108 when measuring time in months, which we should be doing since the payments are monthly.




The amount of principal repaid during the third 3-year period is the difference between the balance of the loan immediately after the payment at time 72 and the balance of the loan immediately after the payment at time 108. Symbolically, we write this as $\sum_{i=73}^{108} P_i = P_{73} + P_{74} + \cdots + P_{108} = B_{72} - B_{108}$.

Although generally not true, in this case we enough information to calculate the outstanding balances either prospectively or retrospectively. E.g., retrospectively, we have $B_{72} = 200000(1.005)^{72} - Rs_{\overline{72}|0.005}$. Notice that we have most everything already stored in the TVM keys because we just calculated the payment $R$. All we need to do is change [N] to 72 and then [CPT] [FV]. We get $B_{72} = 182{,}795.91$. Prospectively, we recognize that immediately after the payment at time 72, we have 360 – 72 = 288 future payments remaining. Then $B_{72} = Ra_{\overline{288}|0.005} = 182{,}795.91$, as before. Likewise, we get $B_{108} = 171{,}580.34$. Therefore the amount of prinicipal repaid during the third 3-year period is 182,795.91 – 171,580.34 = 11,216.

(c) The amount of interest paid in the 1st year is the difference between the total amount paid during the 1st year and the amount of principal repaid during the 1st year. Since the payments are monthly, the total amount paid during the *any* year is $12R$. Similar to part (b), the amount of principal repaid during the 1st year is $B_0 - B_{12} = 2{,}456.02$. Therefore, the amount of interest paid during the 1st year is $12R - (B_0 - B_{12}) = 11{,}933$.

(d) This is done exactly like part (c). The answer is 457. The point of including this problem is to illustrate the fact that early in an amortization of a loan, most of the payments are going towards paying interest, whereas later in the process most of the payments are going towards repaying principal.

4. There are $4n$ quarterly payments of 1000 and the qeir is 2.25%. So $30000 = 1000a_{\overline{4n|}.0225}$ and by TVM we have $4n = 50.5+$. This means it takes 50 quarterly payments of 1000, plus an additional payment, to repay the loan in full. We determine the balance immediately after the 50th payment using the retrospective method by just changing [N] to 50 and then [CPT] [FV]. We get $B_{50} = 503.77$.

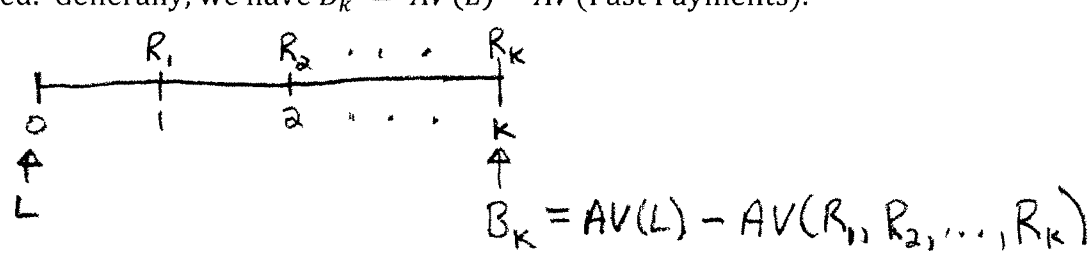
(a) If the final payment is larger than the regular payments of 1000, then we are making the additional payment to repay the loan in full at the time of the last payment of 1000. In this case, we would make an additional payment of 503.77 at the time of our 50th (and last) regular payment of 1000. So the amount of the final payment is 1503.77.

(b) If the final payment is smaller than the regular payments of 1000, then we are making the additional payment one period after the last regular payment. The amount of this additional payment would be the balance immediately after the last regular payment, accumulated with interest. In this case, we make a final payment of 503.77(1.0225) = 515.11 one quarter after the last regular payment of 1000.

5. We have 120 monthly payments, an meir of 0.25%, and we're given $P_{12} = 334.05$. Therefore, $P_1 = 334.05v_{0.0025}^{11}$ and $L = P_1 s_{\overline{120|}0.0025} = 334.05v^{11}s_{\overline{120|}} = 45{,}416$.

Alternatively, we could use the fact that $P_{12} = 334.05 = Rv^{121-12}$ to get $R$ and then use $L = Ra_{\overline{120|}}$ to get $L$.

6. Since we don't know the total number of payments required to pay off the loan, this is a problem where using the retrospective method of determining loan balances is preferred. Generally, we have $B_k \overset{Ret}{=} AV(L) - AV(\text{Past Payments})$.



(a) The outstanding principal is another term for the outstanding balance. I.e. we are to determine $B_5$. We have $AV(L) = 50000(1.0506)^5 = 63{,}996.61$ and $AV(\text{Past Payments}) \overset{VEP}{=} 2000(1.0506)^4 + 2000(1.02)(1.0506)^3 + \cdots$ (Note this sum is geometric with 5 terms and common ratio $\frac{1.02}{1.0506} < 1$.) Since $\frac{1.0506}{1.02} = 1.03$ we get $AV(\text{Past Payments}) = 2000(1.0506)^4\ddot{a}_{\overline{5}|.03} = 11{,}493.56$. Then $B_5 = 52{,}503$.

(b) Using $B_k^{before}$ to denote the balance immediately *before* the $k^{\text{th}}$ payment, we have two ways to determine $B_k^{before}$; namely, $B_k^{before} = B_{k-1}(1+i)$ or $B_k^{before} = B_k + R_k$. These should be intuitive. The first says to accumulate the balance immediately after the prior payment, and the second says to take the balance immediately after the payment at time $k$ and add back in the payment made at time $k$.

Since we've done part (a), the easiest way to proceed to determine the outstanding balance immediately *before* the 6th payment is to use the first method above. We get $B_6^{before} = B_5(1+i) = 52503(1.0506) = 55{,}160$.

Using the second method, $B_6 \overset{Ret}{=} 50000(1.0506)^6 - 2000(1.0506)^5\ddot{a}_{\overline{6}|0.03} = 52{,}951.55$, and $R_6 = 2208.16$, and so $B_6^{before} = 52{,}951.55 + 2208.16 = 55160$.

Notice that the balance immediately after the 6th payment (52,952) is *more than* the balance immediately after the 5th payment (52,503). The reason for this is that we are initially in what's called a **negative amortization**. This means the amount of the initial payments are not enough to cover the interest due, and so the difference is added to the balance of the loan. However, since the payments are increasing geometrically, they will eventually exceed the amount of the interest due, and at that point the loan balances will begin to decrease.

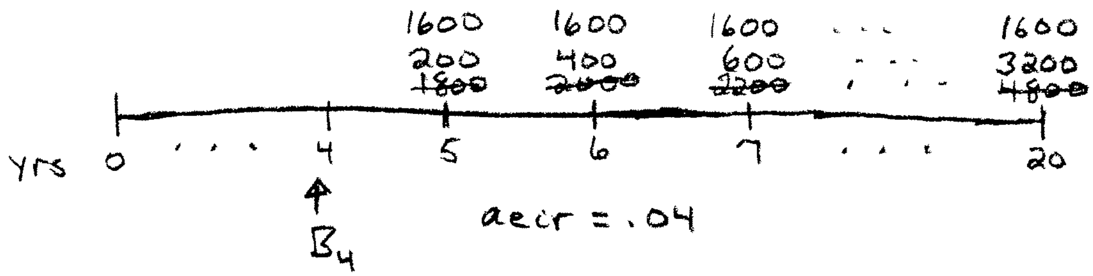
(c) We first determine $B_9$ and then $I_{10} = iB_9$. As in part (a) we get $B_9 \overset{Ret}{=} 50000(1.0506)^9 - 2000(1.0506)^8\ddot{a}_{\overline{9}|0.03} = 54160$. Therefore $I_{10} = (0.0506)(54160) = 2740$.

(d) Since we've just calculated $I_{10}$, the easiest way to determine the principal repaid in the 10th installment is to use $P_{10} = R_{10} - I_{10}$ We have $I_{10} = 2740$, and since $R_{10} = 2000(1.02)^9 = 2390$ then $P_{10} = 2390 - 2740 = -350$.

Note that the payment of 2390 is not enough to cover the interest due of 2740; we're still in a negative amortization. We must add the difference to the balance just after the 9th payment to get the balance just after the 10th payment. Symbolically, we have $B_{10} = B_9 - P_{10} = 54160 - (-350) = 54510$. Of course rewriting this last equation we see that we could have used $P_{10} = B_9 - B_{10}$ to determine $P_{10}$ but what we did was easier.

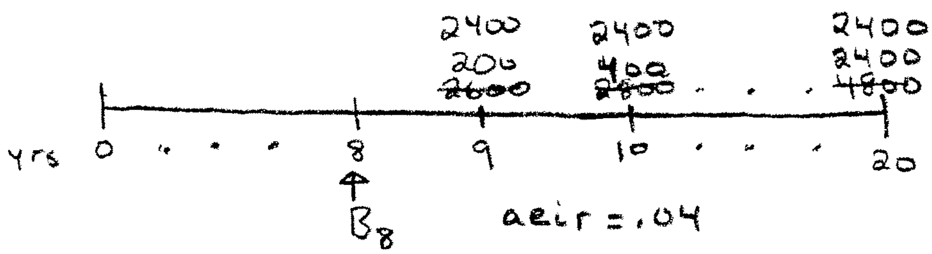
7. The last problem was one in which the retrospective method of determining loan balances was preferable. In this problem, since we don't know the amount of the loan, but rather the number of and amounts of the payments, the prospective method of determining the balances is preferable in this problem. Generally, we have $B_k \overset{Pro}{=} PV(\text{Remaining Payments})$.

(a) Immediately after the 4th payment, there are 16 future payments that increase arithmetically with a common difference of 200. We can write the payments as:




Then $B_4 \overset{Pro}{=} 200(Ia)_{\overline{16|}.04} + 1600a_{\overline{16|}.04}$ =36,522.96.

(b) We know $I_9 = i \cdot B_8$ and we determine $B_8$ as above. The timeline is:




Then $B_8 \overset{Pro}{=} 200(Ia)_{\overline{12|}.04} + 2400a_{\overline{12|}.04} = 33{,}850.74$ and we get $I_9 = .04(33850.74) = 1{,}354.03$.

(c) The second 4-year period extends from time 4 to time 8. We've seen earlier that we can determine the amount of *principal repaid* during this time period by evaluating the difference between the balances at times 4 and 8. Then the amount of interest paid during the period will be the difference between the amount of the payments made during the period and the amount of principal repaid during the period. Symbolically we have $\sum_{k=5}^{8} I_k = \sum_{k=5}^{8} R_k - (B_4 - B_8)$.

From parts (a) and (b), we have $B_4 = 36{,}522.96$ and $B_8 = 33{,}850.74$, and so $\sum_{k=5}^{8} R_k = 1800 + 2000 + 2200 + 2400 = 8{,}400$. Therefore

$$\sum_{k=5}^{8} I_k = 8400 - (36522.96 - 33850.74) = 5{,}727.78$$

**Section 2: Loan Repayment – Sinking Fund Method**

IDEA: Each period, pay interest only to the lender, and in a separate account (the sinking fund account), make payments that will *accumulate* to the loan balance. The interest rate charged by the lender is generally different than the interest rate used for the sinking fund account.

The total amount paid at time $k$ is denoted by $R_k$. This amount can be separated into two parts, the amount of the interest payment to the lender, denoted by $R_k^I$, and the amount of the payment made into the sinking fund, denoted by $R_k^{SF}$. So we have

$$R_k = R_k^I + R_k^{SF}$$

Since the sinking fund deposits accumulate to the loan balance, we generally have $L = AV(R_k^{SF})$.

**Remarks**:

1. If the sinking fund interest rate and the loan interest rate are equal, then the sinking fund and amortization methods are equivalent. (One is indifferent to the two methods.)

2. The net amount of interest paid on the loan during a certain installment (period) equals the difference between the amount of interest paid to the lender and the amount of interest earned in the sinking fund.

**Basic Relationships for Level Payment Sinking Funds**:
Level payments imply $R_k^{SF} = R^{SF}$ for all $k$; that is, the payments are constant.
$i$ = periodic effective interest rate on the loan, charged by the lender
$j$ = periodic effective interest rate used for the sinking fund

$$R^I = L \cdot i$$

$$L = R^{SF} s_{\overline{n}|j} \quad \Longrightarrow \quad R^{SF} = \frac{L}{s_{\overline{n}|j}}$$

Then the total periodic payment is $R = L \cdot i + \frac{L}{s_{\overline{n}|j}} = L(i + \frac{1}{s_{\overline{n}|j}})$. Rewriting, we have

$$L = \frac{R}{i + \frac{1}{s_{\overline{n}|j}}}$$

**Module 3 Section 2 Problems:**

1. A 20-year loan of 100,000 is to be repaid with annual payments using a sinking fund. The lender charges 6% per annum and interest is credited in the sinking fund at 4% per annum.

   (a) Determine the amount of the interest payment to the lender each year.
   (b) Determine the amount of the sinking fund payment each year.
   (c) Determine the total amount of each annual payment.
   (d) Determine the net amount of interest paid during the $13^{th}$ installment.

2. A 10-year loan of 40,000 is to be repaid with monthly payments using the sinking fund method in which interest on the principal is determined using 4% annual effective and deposits into the sinking fund also earn 4% annual effective. Determine the total amount of each monthly payment.

3. A 30-year mortgage of 500,000 is to be repaid with quarterly payments using the sinking fund method in which interest on the principal is determined using 8% compounded quarterly and deposits into the sinking fund earn 6% compounded quarterly. The first quarterly deposit into the sinking fund is $X$ and each subsequent deposit is 25 more than its preceding deposit. Determine the amount of the $45^{th}$ payment.

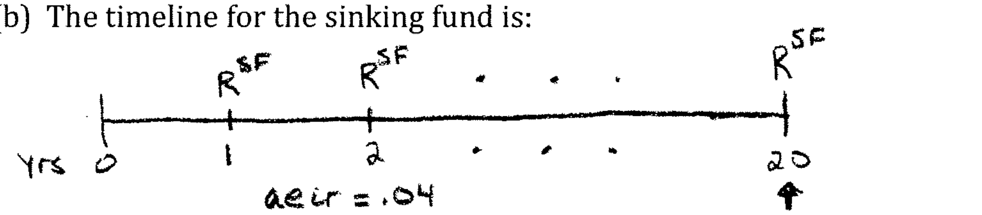
4. A 15-year loan is repaid using the sinking fund method with annual payments to the lender determined using 5% annual effective, and non-level annual payments into the sinking fund that are credited using 4.03% annual effective. The initial sinking fund payment is 1000, and each subsequent payment into the sinking fund is 3% more than its preceding payment. Determine the net amount of interest paid in the $12^{th}$ payment.

**Solutions to Module 3 Section 2 Problems:**

Unless told otherwise, assume deposits into the sinking fund are level.

1. (a) The lender charges $i = .06$ aeir and the loan amount is $L = 100{,}000$. Therefore $R^I = i \cdot L = .06(100000) = 6000$.

(b) The timeline for the sinking fund is:




Therefore, $R^{SF} s_{\overline{20|}.04} = 100{,}000$ and so $R^{SF} = 3358.18$.

(c) The total amount of each payment is $R = R^I + R^{SF} = 9358.18$.

(d) The amount of interest paid to the lender during the 13th year (installment) is the same as during any other year; namely, 6000. However, interest is earned in the sinking fund.

During the 13th year, the amount of interest earned in the sinking fund is the product of 4% (the interest rate earned in the sinking fund) and the balance in the sinking fund at the beginning of the 13th year. The balance at the beginning of the 13th year is the balance immediately after the 12th payment.

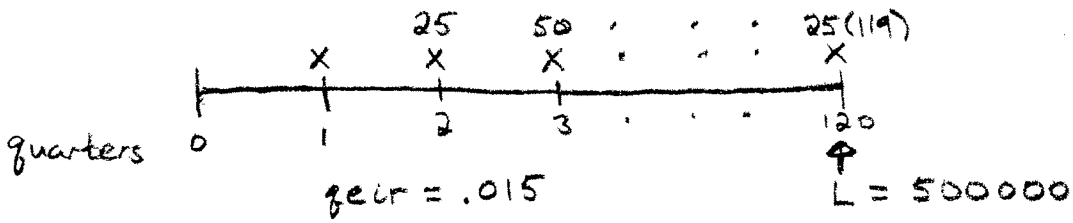
Therefore, the amount of interest earned in the sinking fund during the 13th year is $0.04(Bal_{12}^{SF}) = 0.04\left(R^{SF} s_{\overline{12|}.04}\right) = 2018.37$. Finally, the net amount of interest paid during the 13th installment is 6000 - 2018.37 = 3981.63.

WARNING: Often, students make the mistake of trying to use a prospective calculation for determining the sinking fund balance. Prospective calculations are used for determining loan balances; there is no such analogous method for determining accumulated values of investment accounts.

2. Notice that the lender's interest rate and the sinking fund interest rate are both annual effective, but the payments are monthly. We proceed by first converting the aeir's to meir's. Also note that the lender's rate and the sinking fund rate are the same. Therefore the total payment each period is the same as if we were using an amortization method. I.e., $R = \frac{40000}{a_{\overline{120|}i}}$, where $i = (1.04)^{1/12} - 1$ is the meir equivalent to an aeir of 4%. We get $R = 403.62$.

If we had wanted to, we could have calculated the interest payment and sinking fund payments separately, and added them to get the total payment. We leave it to you to show that if done this way, we would have $R^I = 130.95$ and $R^{SF} = 272.67$.

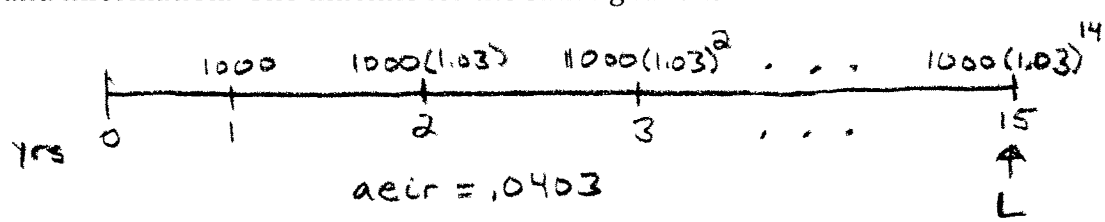
3. First, note the lender is charging a qeir of 2%, and so $R^I = .02(500000) = 10{,}000$. The timeline for the sinking fund is:




We have $500{,}000 = Xs_{\overline{120}|.015} + 25(Is)_{\overline{119}|.015}$ from which we get $R_1^{SF} = X = 446.30$. Then the amount of the 45th sinking fund payment is $R_{45}^{SF} = X + 25(44) = 1546.30$. Therefore the total amount of the 45th payment is $R_{45} = R_{45}^{SF} + R^I = 11{,}546.30$.

4. As we saw earlier, we determine the net amount of interest paid in a period by subtracting the amount of interest earned in the sinking fund for the period from the amount of interest paid to the lender during the period.

In order to determine the interest payment to the lender each period, we must know how much was borrowed. We determine the amount borrowed using the sinking fund information. The timeline for the sinking fund is:




We have $L \overset{VEP}{=} 1000(1.0403)^{14} + 1000(1.03)(1.0403)^{13} + \cdots$ (15 terms). We factor out the first term, getting $L = 1000(1.0403)^{14}(1 + \frac{1.03}{1.0403} + \cdots)$. The last factor is the VEP expression for $\ddot{a}_{\overline{15}|i}$ where $i = \frac{1.0403}{1.03} - 1 = .01$. Therefore, $L = 1000(1.0403)^{14}\ddot{a}_{\overline{15}|.01} = 24{,}348$ and $R^I = .05(24{,}348) = 1217.40$ each year.

The amount of interest earned in the sinking fund for the 12th year is $0.0403(Bal_{11}^{SF})$. As with $L$ above, $Bal_{11}^{SF} \overset{VEP}{=} 1000(1.0403)^{10} + 1000(1.03)(1.0403)^9 + \cdots$ (11 terms), and so $Bal_{11}^{SF} = 1000(1.0403)^{10}\ddot{a}_{\overline{11}|.01} = 15{,}544.86$.

Therefore, the net amount of interest paid for the 12th year is $R^I - .0403(Bal_{11}^{SF}) = 1217.40 - .0403(15544.86) = 590.94$.

**Section 3: Loan Repayment – Adhoc Methods**

Adhoc Methods are loan repayment methods that don't fit into either the amortization or sinking fund categories.

**Module 3 Section 3 Problems:**

1. Interest on a 10-year loan of 25000 is 10% annual effective. The loan is repaid with annual payments at the end of each year, with the first 9 payments equal to 125% of the amount of interest due at the time of the payment, and the final payment at the end of the 10 year period equal to a balloon payment that repays the loan entirely. Determine the amount of the balloon payment.

2. A 10-year loan of 8000 is repaid with annual payment. Each payment consists of a principal payment of 800, plus interest in the unpaid balance using 6% annual effective. When the lender receives a payment, the lender reinvests the payment in an account that pays 4% annual effective. Determine the amount the lender has in the reinvestment account immediately after the last loan payment.

**Solutions to Module 3 Section 3 Problems:**

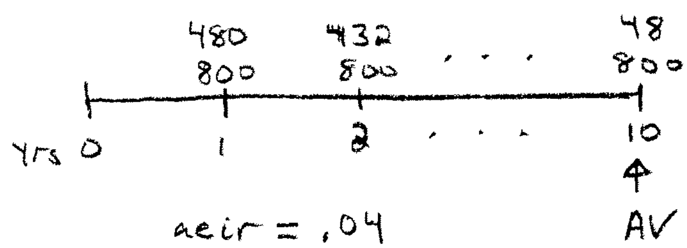
1. For loans in which payments are equal to $X\%$ of the interest due, we develop a pattern for the outstanding balances. Generally, we have

$B_1 = B_0(1+i) - R_1 = B_0(1+i) - 1.25I_1 = B_0(1+i) - 1.25iB_0 = B_0(1-.25i)$. The last equality follows from factoring out $B_0$ from the two terms in the expression on the left side of the equal sign, and then simplifying. Likewise,

$B_2 = B_1(1+i) - R_2 = B_1(1+i) - 1.25I_2 = B_1(1+i) - 1.25iB_1 = B_1(1-.25i)$. Substituting $B_1 = B_0(1-.25i)$, we have $B_2 = B_0(1-.25i)^2$, and continuing, we get $B_9 = B_0(1-.25i)^9$.

Note that the amount of the balloon payment will be the balance just after the 9th payment, accumulated one year at 10% annual effective. Therefore, the final payment is $P = B_0(1-.25i)^9(1+i) = 25000(.975)^9(1.1) = 21896.48$.

2. The amount of the first payment is $R_1 = P_1 + I_1 = 800 + .06(8000) = 800 + 480$. Then $B_1 = B_0 - P_1 = 8000 - 800 = 7200$, and so the amount of the second payment is $R_2 = P_2 + I_2 = 800 + .06(7200) = 800 + 432$. Continue. The timeline of the payments is




The amount the lender has after the last loan payment is the accumulated value of the above payments using 4% annual effective. Recognizing the payments as a 10-year level annuity with payments of 800, together with a 10-year arithmetically decreasing annuity with common difference equal to 48, we have

$$AV = 800s_{\overline{10|}.04} + 48(Ds)_{\overline{10|}.04} = 12{,}960.49$$

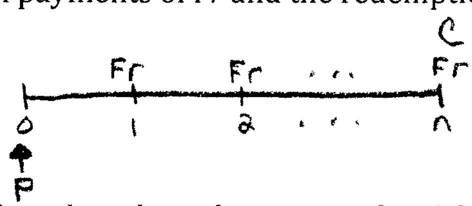
**Section 4: Bond Introduction**

Bonds are nothing more than loans, but from a different perspective. When we purchase a bond, we are lending money to the issuer of the bond. So we are taking the perspective of the lender rather than the borrower.

**Terminology and Notation**:

$P$ – price of the bond (the amount we are lending)
$i$ – yield rate (as a periodic effective interest rate)
$F$ – face value (or par value)
$r$ – coupon rate (as a periodic effective interest rate)
$Fr$ – coupon amount
$n$ – number of coupons
$C$ – redemption value

When we purchase a bond, we pay the price $P$ today in exchange for receiving the periodic coupon payments of $Fr$ and the redemption value of C at time $n$. The timeline is:




If the price $P$ is less than the redemption value $C$ (i.e. $P < C$), we say the bond is bought at a **discount**. The amount of discount is $C - P$. If the price $P$ is more than the redemption value $C$ (i.e. $P > C$) we say the bond is bought at a **premium**. The amount of premium is $P - C$. If the price $P$ is equal to the redemption value $C$ (i.e. $P = C$) we say the bond is bought at a **par**.

BEWARE:

1. Most bonds are "redeemable at par". This means the redemption value $C$ is equal to the face value $F$ (i.e. $C = F$). Note the difference between "bought at par" and "redeemable at par". Unless told or implied otherwise, you may assume the bond is redeemable at par.
2. In the problems, the stated rate associated to the coupons is a nominal rate. For example, if the problem states we have a 1000 par value bond with 8% semiannual coupons, then the 8% is a nominal rate compounded semiannually. The coupon rate, $r$, would be 4% in this case, and so the amount of each coupon would be 1000(0.04) = 40.

**Pricing Formulas**:

$P = Fra_{\overline{n|}i} + Cv_i^n$ (Standard Pricing Formula)
$P = C + (Fr - Ci)a_{\overline{n|}i}$ (Premium/Discount Formula)

**Module 3 Section 4 Problems:**

1. A 10-year 1000 face value bond with 8% annual coupons is bought to yield 6% annual effective. The purchase price is 1259. Determine the redemption value to the nearest dollar.

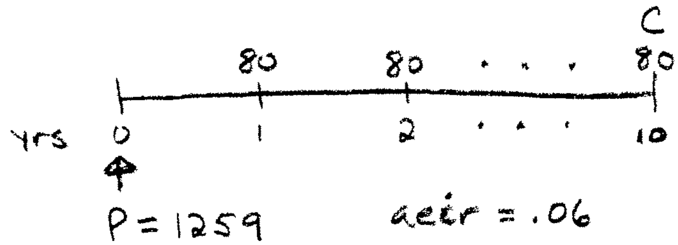
2. A 20-year 1000 par value bond, redeemable at par, with 7% semiannual coupons is bought for 901. Determine the annual effective yield for the bond.

3. A 20-year 100 par value bond with 8% quarterly coupons is bought to yield 6% compounded quarterly. Determine whether the bond is bought at a premium or discount, and determine the amount of premium or discount.

4. An $n$-year bond, redeemable at par, with 6% coupons payable semiannually, is bought to yield 7% compounded semiannually. Determine whether the bond is bought at a premium or discount.

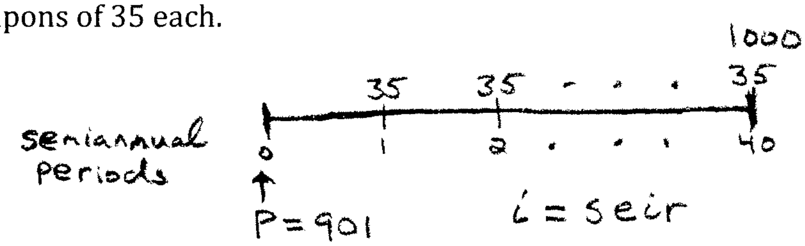
5. A bond has quarterly coupons of 18 and is redeemable for 1200. Determine the yield rate, as a nominal rate compounded quarterly, at which the bond would be bought at par.

**Solutions to Module 3 Section 4 Problems:**

1. 

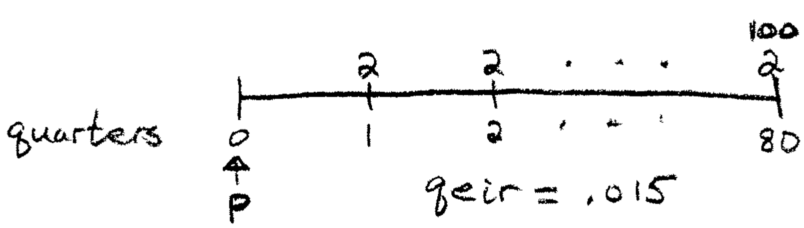

From the basic pricing formula, we have $1259 = 80a_{\overline{10}|.06} + Cv_{.06}^{10}$. Since there are 10 payments to discount and the redemption value is discounted for 10 periods, both at the same eir, then we can determine $C$ in one step as follows: 10 [N] 6 [I/Y] 1259 [PV] 80 [+/−] [PMT] [CPT] [FV]. The redemption value is $C = 1200$. Notice the calculator displays a negative 1200, since the +/- sign of the redemption value should agree with the +/- sign of the payment, and should be opposite in sign to the +/- sign of the price.

2. First, note that the 7% semiannual coupon rate is nominal, which gives a 3.5% semiannual effective coupon rate. So $r = 0.035$, and there are 40 semiannual coupons of 35 each.




The basic pricing formula is $901 = 35a_{\overline{40}|i} + 1000v^{40}$, which by using TVM gives $i = .04$. Since time is being measure in semiannual periods, this is an seir. We want the *annual* yield, and so we convert the 4% seir to its equivalent aeir; namely $1.04^2 - 1 = .0816$. The bond was bought to yield 8.16% annual effective.

3. The coupon rate is $r = .02$ quarterly effective, the yield rate is $i = .015$ quarterly effective, and since we cannot proceed otherwise, we assume the bond is redeemable at its face value of 100. The timeline for this bond is




Using the basic pricing formula, we have $P = 2a_{\overline{80}|.015} + 100v^{80} = 123.20$. Therefore the bond is bought at a premium of 23.20.

4. The coupon rate is $r = .03$ semiannual effective and the yield rate is $i = .035$ semiannual effective. The easiest way to determine whether a bond is bought at a premium or at a discount is to use the premium/discount pricing formula, $P = C + (Fr - Ci)a_{\overline{n}|i}$. The bond is bought at a premium if $P > C$, the bond is bought at a discount if $P < C$, and the bond is bought at par if $P = C$. Rewriting the premium/discount formula as $P - C = (Fr - Ci)a_{\overline{n}|i}$, we have

$Fr > Ci$ is equivalent to the bond being bought at a premium.
$Fr < Ci$ is equivalent to the bond being bought at a discount.
$Fr = Ci$ is equivalent to the bond being bought at par.

This bond is redeemable at par, which means $C = F$. Therefore, since $r < i$, this bond is bought at a discount.

5. See the solution to #4. We have $Fr = 18$ and $C = 1200$. So the bond is bought at par if $i = \frac{18}{1200} = .015$ quarterly effective. The equivalent nominal rate compounded quarterly is $i^{(4)} = 4i = .06$. This bond is bought at par if it is bought to yield 6% compounded quarterly.

## Section 5: Bond Amortization Schedule

**Notation**:

We use the same symbols in a bond amortization table as we do in a loan amortization table. In fact, the same relationships hold in a bond amortization table as in a loan amortization table, with the only differences being in terminology. We have

$B_k$ – book value (or amortized value) immediately after the $k^{\text{th}}$ coupon payment (this is like the loan balance)

$I_k$ – Amount of the $k^{\text{th}}$ coupon that represents the amount of interest earned (instead of the amount or interest paid)

$P_k$ – Amount of the $k^{\text{th}}$ coupon that represents principal adjustment (instead of principal repaid)

**Basic Relationships for Bond Amortizations**:

$B_0 = P = Fra_{\overline{n}|i} + Cv^n$ = the price of the bond
$B_n = C$ = the redemption value of the bond.
(Think of the redemption value as being paid one second after time *n*.)

$B_k \overset{Pro}{=} Fra_{\overline{n-k}|i} + Cv^{n-k}$ ( = PV of future coupons and redemption value, using *i*)
(We'll see a retrospective formula later.)

Just as with loans, $I_k = i \cdot B_{k-1}$ and then $P_k = Fr - I_k$

These relationships are captured in a **Bond Amortization Table**:

| Time | Coupon | Interest Earned | Principal Adjustment | Book Value (Amortized Value) |
|---|---|---|---|---|
| 0 | | | | $P = B_0 = Fra_{\overline{n}|} + Cv^n$ |
| 1 | Fr | $I_1 = i \cdot P$ | $P_1 = Fr - I_1$ | $B_1 = Fra_{\overline{n-1}|} + Cv^{n-1}$ |
| 2 | Fr | $I_2 = i \cdot B_1$ | $P_2 = Fr - I_2$ | $B_2 = Fra_{\overline{n-2}|} + Cv^{n-2}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $n$ | Fr | $I_n = i \cdot B_{n-1}$ | $P_n = Fr - I_n$ | $B_n = C$ |

**Remarks about this table**:

1. $\{P_1, P_2, \cdots, P_n\}$ is a geometric sequence with common ratio $r = 1 + i$.

2. We can relate the book value at time $k$ to the book value at time $m$ ($k < m$) as:

   $B_k = Fra_{\overline{m-k|}} + B_m v^{m-k}$ (Note that this is a one-step TVM calculation.)

   As written, this equation has a valuation date at time $k$. Multiplying both sides by $(1+i)^{m-k}$ and rearranging terms gives the time $m$ equation $B_m = B_k(1+i)^{m-k} - Rs_{\overline{m-k|}}$. With $k = 0$, this is the prospective method of determining the book value.

3. As a special case of the previous remark, we can calculate book values at neighboring times in two ways:

   $$B_{k+1} = B_k(1+i) - Fr$$

   or

   $$B_{k+1} = B_k - P_{k+1}$$

   The above remarks are identical to the remarks made about a level payment loan amortization table. The next remarks are specific to a bond amortization.

4. Recall that the bond is bought at a premium if $P > C$. In this case, since $B_0 = P$ and $B_n = C$, the book values are systematically *decreasing* from time 0 to time $n$. We say the bond is **written down** and each of the $P_k$ values is a positive value that we call the **amortization of premium**, or **amount of write-down**, for period (or installment) $k$.

5. Recall that the bond is bought at a discount if $P < C$. In this case, since $B_0 = P$ and $B_n = C$, the book values are systematically *increasing* from time 0 to time $n$. We say the bond is **written up** and each of the $P_k$ values is a *negative* value. Exams may refer to the absolute value of the $P_k$ values as the **accumulation of discount**, or **amount of write-down**, for period (or installment) $k$. Note that accumulation of discount or amount of write-down implies the bond was bought at a discount, and so the $P_k$'s are *negative*.

6. Regardless of whether the bond is bought at a premium or at a discount, $\sum_{i=1}^{n} P_i = P_1 + P_2 + \cdots + P_n = P - C$. Notice that this value is positive if the bond is bought at a premium, but is negative if the bond is bought at a discount.

**Module 3 Section 5 Problems:**

1. A 10-year 10,000 face value bond with 5% annual coupons is bought to yield 6% compounded annually. Determine the book value immediately after the $4^{th}$ coupon.

2. A 10-year 10,000 face value bond with 5% annual coupons is bought to yield 6% compounded annually. Determine the book value immediately before the $4^{th}$ coupon.

3. A bond with semiannual coupons of 5 is bought to yield 4% compounded semiannually. The book value at the end of the $6^{th}$ year is 162.48. Determine the price paid for the bond.

4. A bond that was bought to yield 3% annual effective has annual coupons of 50. The book value is 1400 immediately after the seventh coupon is paid. Determine the book value immediately after the tenth coupon is paid.

5. A bond for which the book value is 937 immediately after the $k^{th}$ coupon is paid has an accumulation of discount of 8 during period $(k + 1)$. Determine the book value of the bond immediately after the coupon at time $(k + 1)$ is paid.

6. A 10-year 1000 par value bond with 8% annual coupons is bought to yield 6% annual effective. Determine the amount of interest earned during the $3^{rd}$ year.

7. A bond has a book value, immediately after the $8^{th}$ coupon is paid, of 878. The coupons are 35 each, and the principal adjustment for the $9^{th}$ installment is 8.66. Determine the periodic effective yield rate for which the bond was bought.

8. A 20-year 1000 face value bond with 7% semiannual coupons is bought for 901. Determine the amount of interest earned during the $12^{th}$ year.

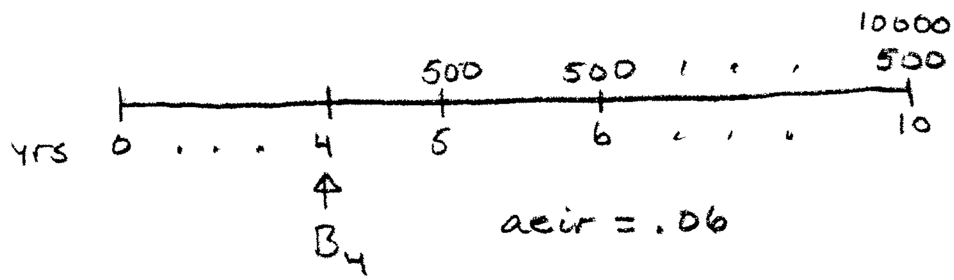
9. A 15-year 1000 face value bond with 7% annual coupons and redemption value of 1250 is bought to yield 5% annual effective. Determine the amount of principal adjustment for the eighth year.

10. A 10-year bond with semiannual coupons is bought to yield 8% compounded semiannually. The amortization of premium for the $8^{th}$ installment is 6.07. Determine the amount of premium for which this bond was bought.

11. A 6-year bond, redeemable at 1500, with quarterly coupons is bought to yield 8% compounded quarterly. The amount of write-up for the $4^{th}$ installment is 3.57. Determine the price paid for the bond.

**Solutions to Module 3 Section 5 Problems:**

1. The coupons are 500 each and we assume the bond is redeemable at its face value of 10000. The most direct way to determine the book value immediately after the 4th coupon is paid is to use the prospective method of determining book values.




There are 6 remaining coupons and the redemption value still to be paid, and so we have $B_4 = 500a_{\overline{6}|.06} + 10000v^6 = 9508.27$.

2. This is the same bond as in the previous problem. If we want the book value immediately before the 4th coupon is paid, then we include the 4th coupon as a future coupon in our prospective formula. Since we have $B_4 = 9508.27$ from above, we get $B_4^{before} = Fr + B_4 = 500 + 9508.27 = 10{,}008.27$.

We could also use the fact that $B_4^{before} = B_3(1+i)$. You should check that you'll get the same answer using this method. Notice that the book values are not continuous over time, but that there are jump discontinuities at the time at which the coupons are paid. In fact the size of the jump discontinuity is the amount of the coupon payment.

WARNING: A common mistake is to try to determine the book value immediately before a coupon is paid by using annuity due notation. This will be wrong because the number of remaining coupons will not match the number of periods the redemption value is being discounted.

3. The price is the book value at time 0; i.e. $P = B_0$. We are given the book value at the end of the 6th year, but since coupons are paid semiannually, this corresponds to time 12. The semiannual effective yield on the bond is $i = .02$, and so we have $P = 5a_{\overline{12}|.02} + 162.48v^{12} = 180.99$.

4. Generally we have, $B_7 = Fra_{\overline{3}|i} + B_{10}v^3$. Here, we get $1400 = 50a_{\overline{3}|.03} + B_{10}v^3$ which gives $B_{10} = 1370.64$. (One-Step TVM: 3 [N] 3 [I/Y] 1400 [PV] 50 [+/−] [PMT] [CPT] [FV].)

5. Since we're told there's an accumulation of discount, then we know the bond is being written up; i.e. the book values are increasing. So $B_{k+1} = 937 + 8 = 945$. (Generally, we have $B_{k+1} = B_k - P_{k+1}$, but since the bond is being written up, $P_{k+1} = -8$.)

6. $I_3 = i \cdot B_2$. We have $B_2 = 80a_{\overline{8}|.06} + 1000v^8 = 1124.20$, and since $i = .06$, we have $I_3 = 0.06 \cdot (1124.20) = 67.45$.

7. We have $I_9 = Fr - P_9 = 35 - 8.66 = 26.34$, but we also have $I_9 = i \cdot B_8 = 878i$. Therefore, $26.34 = 878i \Rightarrow i = .03$.

8. Be careful to note that coupons of 35 are paid semiannually. So the amount of interest earned during the 12th year equals the sum of the amounts of interest earned during the 23rd and 24th periods. We can determine this sum as follows: $I_{23} + I_{24} = i \cdot B_{22} + i \cdot B_{23} = i \cdot (B_{22} + B_{23})$. For the semiannual effective yield, $i$, $901 = 35a_{\overline{40}|i} + 1000v^{40} \Rightarrow i \approx 0.04$. Then the book values are determine as
$901 = 35a_{\overline{22}|i} + B_{22}v^{22} \Rightarrow B_{22} = 936.68$, and
$901 = 35a_{\overline{23}|i} + B_{23}v^{23} \Rightarrow B_{23} = 939.15$, using unrounded $i$. Therefore,
$I_{23} + I_{24} = i \cdot (B_{22} + B_{23}) = .04(936.68 + 939.15) = 75.03$.

9. Each coupon is 70, and so $P_8 = 70 - I_8 = 70 - i \cdot B_7$. We're given $i = .05$, and $B_7 = 70a_{\overline{8}|i} + 1250v^8 = 1298.47$. Therefore, $P_8 = 5.08$.

10. There are 20 coupons. The amount of premium is $P - C = \sum_{k=1}^{20} P_k$. We're given $P_8 = 6.07$ and the semiannual effective yield is $i = 0.04$. So $P_1 = P_8 v_{.04}^7 = 6.07v^7$. Note $\sum_{k=1}^{20} P_k = P_1 + P_2 + \cdots + P_{20} = P_1(1 + (1+i) + \cdots + (1+i)^{19}) = P_1 s_{\overline{20}|}$. So, $P - C = \sum_{k=1}^{20} P_k = 6.07v^7 s_{\overline{20}|.04} = 137.36$.

11. Since the amount of write-*up* for the 4th installment is 3.57, we have $P_4 = -3.57$. The quarterly effective yield is $i = .02$ and so $P_1 = P_4 v_{.02}^3 = -3.57v^3$. Then as in the last problem, $P - C = \sum_{k=1}^{24} P_k = -3.57v^3 s_{\overline{24}|.02} = -102.34$, and since $C = 1500$, we have $P = 1500 - 102.34 = 1397.66$.

## Section 6: Callable Bonds

A callable bond is a bond in which the issuer reserves the right to redeem the bond at different discrete times, possibly for different redemption values. When a callable bond is bought, no one knows when the bond will be called, however it can only be called at the times agreed upon at issue.

Callable bond problems will be obvious by the wording of the problem. Unless told that the bond is callable within the wording of the problem, assume all bonds are held to maturity and redeemed for the stated redemption value.

The most common questions asked with callable bonds involve either

(i) determining the maximum price that can be paid for the bond to guarantee a certain yield rate, or

(ii) determining the minimum yield rate for a bond that was bought at a certain price.

The recommended strategy for solving callable bond problems is to create a two-column table with the first column representing the time at which the bond can be called (or redeemed).

For type (i) problems, the second column is the corresponding price in order to receive the desired yield rate, keeping in mind the redemption value may be different at the different redemption dates. Then think through what happens when the bond is bought for the prices in the table but redeemed at other times in the table, and answer the question accordingly.

For type (ii) problems, the second column of the table is the yield rate that would produce the given price, again keeping in mind the redemption value may be different at the different redemption dates. Answer the question accordingly.

**Module 3 Section 6 Problems:**

1. A 1000 face value 20-year callable bond with 5% annual coupons is selling for 1150. The bond can be redeemed at the end of 18 years for 950, at the end of 19 years for 975, or at the end of 20 years for 1000. Determine the minimum annual yield rate that a buyer will earn on this bond.

2. A 1000 face value 20-year callable bond, redeemable at 1200, with 5% annual coupons can be redeemed at the end of year 18, 19, or 20. Determine the maximum price a buyer is willing to pay in order to earn an annual yield of at least 3%.

3. A 1000 face value 20-year callable bond with 4% annual coupons can be redeemed according to the following schedule:

   1000 at the end of years 12 through 14,
   1025 at the end of years 15 through 17,
   975 at the end of years 18 through 20.

   Determine the maximum price a buyer is willing to pay in order to earn an annual yield of at least 4%.

4. A 1000 face value 10% annual coupon bond is redeemable as follows:

   1100 at the end of years 15, 16, or 17
   1000 at the end of years 18, 19, or 20.

   A buyer pays 1500 for this bond. Determine the buyer's minimum annual yield.

**Solutions to Module 3 Section 6 Problems:**

1. We create the following table:

| Call Time (in years) | Annual Yield Rate |
|---|---|
| 18 | $1150 = 50a_{\overline{18}\vert i} + 950v^{18} \Rightarrow i = 0.03647$ |
| 19 | $1150 = 50a_{\overline{19}\vert i} + 975v^{19} \Rightarrow i = 0.03786$ |
| 20 | $1150 = 50a_{\overline{20}\vert i} + 1000v^{20} \Rightarrow i = 0.03905$ |

The minimum annual yield rate a buyer of this bond will earn is 3.647%. The buyer earns this yield if the bond is redeemed after 18 years. If the bond is redeemed at time 19 or 20, then the buyer will earn an annual yield of 3.786% or 3.905%, respectively. In either case the annual yield would be greater than 3.657%.

2. We create the following table:

| Call Time (in years) | $P = P(.03) =$ Price to yield 3% annual effective |
|---|---|
| 18 | $P = 50a_{\overline{18}\vert .03} + 1200v^{18} = 1392.55$ |
| 19 | $P = 50a_{\overline{19}\vert .03} + 1200v^{19} = 1400.53$ |
| 20 | $P = 50a_{\overline{20}\vert .03} + 1200v^{20} = 1408.28$ |

Be careful with this type of problem. We are asked to find the maximum price in order to achieve an annual yield of at least 3%. It is tempting to use the maximum price from the table values, but let's see what's wrong with that logic.

Suppose a buyer pays 1408.28 for the bond. Again, at the time of purchase, no one knows when the bond is going to be redeemed. If the bond issuer happens to redeem the bond at the end of 18 years, then as seen in the table, the buyer should have paid 1392.55. The buyer paid too much, and so the buyer's annual yield would be less than 3%. In fact, we can determine the buyer's annual yield in this case as $1408.28 = 50a_{\overline{18}|i} + 1200v^{18} \Rightarrow i = 0.02914$, which is less than 3% as expected.

The above paragraph illustrates that we should choose the minimum price in the table. By choosing 1392.55, then regardless of when the bond is called, the buyer will earn an annual yield of at least 3%. For example, the three cases are:

(i) the bond is called at the end of year 18, in which case the buyer's annual yield is 3%
(ii) the bond is called at the end of year 19, in which case we calculate the buyer's annual yield as $1392.55 = 50a_{\overline{19}|i} + 1200v^{19} \Rightarrow i = 0.03042 > 0.03$, or
(iii) the bond is called at the end of year 20, in which case we calculate the buyer's annual yield as $1392.55 = 50a_{\overline{20}|i} + 1200v^{20} \Rightarrow i = 0.03080 > 0.03$.

Our answer is that the maximum price the buyer should pay in order to guarantee an annual yield of at least 3% is 1392.55.

One final observation is that if the redemption values are level, as they are in this problem, then we only need to check the extreme values. For example, in this problem we didn't need to include the row in the table corresponding to time 19.

3. According to the last paragraph, we don't need rows in our table corresponding to times 13, 16, and 19. We create the following table:

| Call Time (in years) | $P = P(.04) =$ Price to yield 4% annual effective |
|---|---|
| 12 | $P = 40a_{\overline{12}\rceil.04} + 1000v^{12} = 1000$ |
| 14 | $P = 40a_{\overline{14}\rceil.04} + 1000v^{14} = 1000$ |
| 15 | $P = 40a_{\overline{15}\rceil.04} + 1025v^{15} = 1013.88$ |
| 17 | $P = 40a_{\overline{17}\rceil.04} + 1025v^{17} = 1012.83$ |
| 18 | $P = 40a_{\overline{18}\rceil.04} + 975v^{18} = 987.66$ |
| 20 | $P = 40a_{\overline{20}\rceil.04} + 975v^{20} = 988.59$ |

The maximum price a buyer of this bond should pay in order to guarantee an annual yield of at least 4% is 987.66. (If any higher price is paid, and the bond is ultimately called at the end of year 18, then the buyer's annual yield will be less than 4%.)

4. We only need rows in our table corresponding to times 15, 17, 18, and 20. We create the following table:

| Call Time (in years) | Annual Yield Rate |
|---|---|
| 15 | $1500 = 100a_{\overline{15}\rceil i} + 1100v^{15} \Rightarrow i = 0.05474$ |
| 17 | $1500 = 100a_{\overline{17}\rceil i} + 1100v^{17} \Rightarrow i = 0.05696$ |
| 18 | $1500 = 100a_{\overline{18}\rceil i} + 1000v^{18} \Rightarrow i = 0.05540$ |
| 20 | $1500 = 100a_{\overline{20}\rceil i} + 1000v^{20} \Rightarrow i = 0.05734$ |

The minimum annual yield rate a buyer of this bond will earn is 5.474%.